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EXPLOITATION OF MULTIPLE SOLUTIONS OF THE NAVIER-STOKES EQUATIONS TO ACHIEVE RADICALLY IMPROVED FLIGHT

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by

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It is known that the nonlinear Navier-Stokes equations will model most fluid flow of aeronautical interest. The existence and uniqueness of the solutions to the Navier-Stokes equations have not been proven although it is known that in certain cases only the most stable solution is obtained. This present work is concerned with identifying multiple solutions of the Navier-Stokes equations for transonic flow. The objective is to exploit the existence of these solutions rather than avoid them as has been the custom in the past. The present work has shown that the cause of multiple solutions in potential flow is a bifurcation of solutions at a specific Mach number distribution; airfoils can be designed to give such a distribution. It is also found that the presence of entropy and vorticity do not affect the occurrence of phantom solutions. A physical example of a phantom solution is explained by a study of the potential phantom solutions.

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EXPLOITATION OF MULTIPLE SOLUTIONS OF THE NAVIER-STOKES EQUATIONS TO ACHIEVE RADICALLY IMPROVED FLIGHT

Objectives of the Research

It is generally accepted that all fluid flow phenomena of practical use, including turbulence, can be represented by the Navier-Stokes equations. The Navier-Stokes equations are a set of coupled nonlinear partial differential equations representing the conservation of mass, momentum, and energy. The nonlinearity of the equations allows the representation of turbulence, shock waves, and slip surfaces. If the equations are Reynolds averaged, then the ability to represent the details of turbulence is lost, but the nonlinearity necessary to represent shock waves and slip surfaces is retained. Little is known about the existence and uniqueness of the solutions to nonlinear equations. A common test of existence is to see if the solutions of the Navier-Stokes equations closely model the behavior observed in wind tunnel tests. It is reasonable to speculate that if the Navier-Stokes equations had several solutions, one of which closely models an observed flow, then the other solutions must be physically realizable, although probably unstable. This raises the intriguing question of whether these realizable but unstable solutions represent a radical change in flow behavior that may be of great benefit in a practical aerodynamic sense.

In the present work, the conventional solution is defined as the solution that would be expected from an extrapolation from previous knowledge and a "phantom" solution is defined as any additional solution.

One type of phantom solution that is found in potential transonic flow theory, a simpler subset of the Navier-Stokes equations, indicates that there are several different shock patterns that satisfy the boundary conditions. The typical example of this type of phantom solution is that of the flow around a symmetric airfoil at zero angle of attack giving lift; the phenomena appears also for conventional lifting airfoils. An example[1] of the variation of lift coefficient, C_L, with angle of attack α, for a RAE2822 airfoil is shown in Figure 1. It may be observed that at α = 0.6 degrees, C_L can have a value of about 0.5 or about 1.4. Phantom solutions have also been found for three-dimensional flows.[2] This phenomena was first reported by Steinhoff and Jameson.[3] These phantom solutions appear when some degree of asymmetry, either in the algorithm or in the initial conditions, is introduced. The asymmetry may also be introduced into the solution process by a physical angle of attack. It appears that there is a certain range of Mach number in which the phantom solution is preferred to the conventional solution. These phantom solutions have, until recently, not been observed in equation sets other than the potential equation; although, it has been speculated by Williams et al[4] and Nixon[5] that they would exist for the Euler equations. Nixon[6] has recently computed phantom solutions for an equation, the "TSD-Euler" equation, which closely resembles the Euler equations. For configurations that would normally give lift, the presence of one of the phantom solutions can increase the lift considerably, by a factor of two or three (see Figure 1). It is therefore a more than academic point to consider whether these phantom solutions can exist in practice.

The present work is concerned with a continuing investigation of the appearance of phantom solutions for the TSD-Euler equation with a view to determining if such solutions can exist for the Navier-Stokes equations and thus indicating that they may be physically realizable. The evidence gained during the present study does indicate that phantom solutions could occur in reality and thus may introduce a new aspect of aerodynamics, namely the exploitation of nonlinear aerodynamic phenomena.

It was anticipated that the tasks that were to be performed during the course of the contract should be as follows:

(a) Study further the occurrence of phantom solutions of the TSD-Euler equation to get a clearer idea of the mechanisms and control of such solutions. Attempt to develop a means of differentiating a phantom solution from a conventional solution.

(b) Extend the XTRAN2L code with boundary layer to treat the TSD-Euler equation. Determine the effect of a boundary layer on the occurrence and control of phantom solutions.

(c) Using the ARC2D code, attempt to reproduce the phantom solutions for the Euler equations.

(d) Using the ARC2D code, attempt to reproduce the phantom solutions for the Navier-Stokes equations.

Status of Research Effort

The TSD-Euler equation, derived in previous work was examined for phantom solutions in an attempt to determine the effect on entropy and vorticity on the appearance of such solutions. It was found that phantom solutions can be found for this equation although for a different range of Mach numbers than for the TSD equation, at least for the airfoils considered. Following earlier work a source of vorticity was introduced on the airfoil surface and this enhanced the probability of the appearance of phantom solutions. Closer study indicated that the dominant effect of the vorticity introduction is to change the effective geometry of the airfoil with the implication that an airfoil could be designed to give a phantom solution to the TSD-Euler equation. The vorticity introduction can be regarded as a method of flow control which mimics the effect of a change in geometry.

A test of when a phantom solution can appear has been developed and it indicates that the dominant feature determining the appearance of phantom solutions is the local Mach number distribution; the entropy or vorticity in the flow field plays only a secondary role. The numerical verification of the test has some problems due principally to the numerical inaccuracies in the numerical solutions of the flow and in the evaluation of the test parameters. The test indicates that a phantom solution can be initiated by designing the airfoil to give a suitable distribution of Mach number in the flow field.

The inference of the study of the TSD-Euler equation is that entropy and vorticity do not inhibit the appearance of phantom solutions but simply change the freestream conditions for which these solutions can appear. Since vorticity does not inhibit the appearance of the phantom solutions it is pertinent to ask if the Navier-Stokes equations have phantom solutions. It is known that for a small Mach number range that the Navier-Stokes equations do have multiple solutions, are symmetric and are oscillatory. This behavior is shown to be explainable by the idea of phantom solutions, although this may not be the only explanation. The multiple solutions of the Navier-Stokes equations are apparent in wind tunnel tests, indicating their realism. The question that now arises is whether phantom solutions can be made to exist stably in a real environment thus introducing a new area of study for aerodynamicists and, possibly, introducing a new mechanism for flight. In order to achieve this the phantom solutions must be stable and some form of flow control is necessary.

If the test derived in this present work can be made more accurate numerically it should be possible to design an airfoil to give phantom solutions over a range of flight conditions. Again, some form of flow control is necessary. If such an airfoil can be designed then phantom solutions to the Navier-Stokes equations may be found, together with some means of stabilizing such solutions. After this stage a wind tunnel test could be conducted. To achieve such an end it is necessary to learn much more about the nature of all phantom solutions, not just those considered here, that is, the transonic solutions. A more complete discussion of this work is contained in the Appendix.

It proved difficult to run the XTRAN2L code with the boundary layer. It was decided not to attempt to compute Euler solutions because of the unrealistic nature of the boundary conditions. Phantom solutions to the Navier-Stokes equations were computed but are unsteady. Efforts to stabilize these solutions failed.

(c) Written Publications

Further Study of Phantom Solutions to the TSD-Euler Equation
(to be submitted for publication in Acta Mechanica)

(d) Personnel

Dr. David Nixon
Dr. Steven C. Caruso
Dr. Mohammad Farshchi

No degrees have been awarded as a result in this work.

(e) Interactions

The work has been discussed with Dr. Joseph Shang, Dr. James Olsen, and Dr. Jan Lee at AFWAL. No other interactions have taken place.

(f) No new discoveries or inventions, patent disclosures or specific applications have stemmed from the research other than those discoveries detailed in this report.

(g) None.

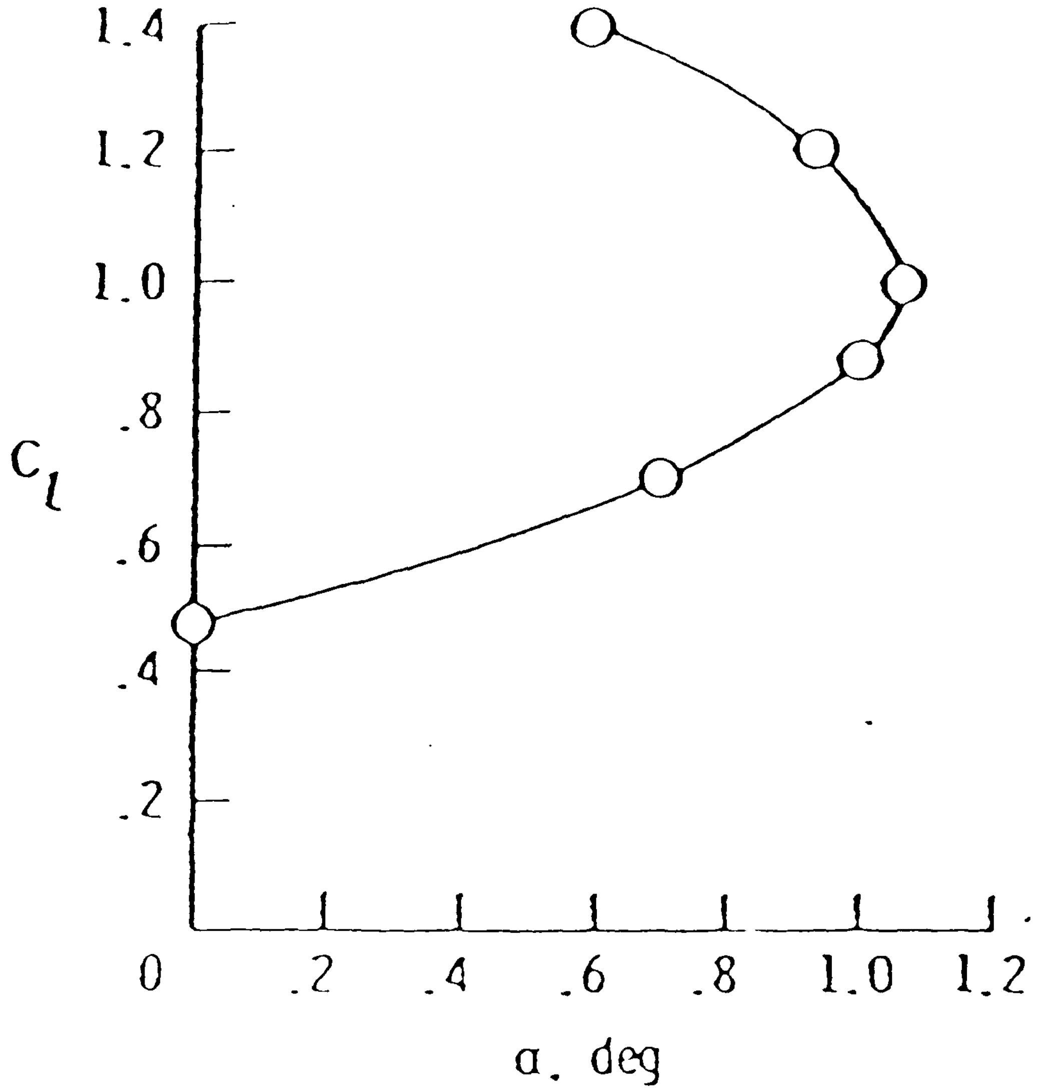

Figure 1.- C_L-α curve obtained with FLO 36 for an RAE 2822 airfoil.

APPENDIX

FURTHER STUDY OF PHANTOM SOLUTIONS TO THE "TSD-EULER" EQUATION

Introduction

It is generally accepted that all fluid flow phenomena of practical use, including turbulence, can be represented by the Navier-Stokes equations. The Navier-Stokes equations are a set of coupled nonlinear partial differential equations representing the conservation of mass, momentum, and energy. The nonlinearity of the equations allows the representation of turbulence, shock waves, and slip surfaces. If the equations are Reynolds averaged, then the ability to represent the details of turbulence is lost, but the nonlinearity necessary to represent shock waves and slip surfaces is retained. Little is known about the existence and uniqueness of the solutions to nonlinear equations. A common test of existence is to see if the solutions of the Navier-Stokes equations closely models the behavior observed in wind tunnel tests. It is reasonable to speculate that if the Navier-Stokes equations had several solutions, one of which closely models an observed flow, then the other solutions must be physically realizable, although probably unstable. This raises the intriguing question of whether these realizable but unstable solutions represent a radical change in flow behavior that may be of great benefit in a practical aerodynamic sense.

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Basic Equations

The TSD-Euler equation [5] is derived from the classic Transonic Small Disturbance (TSD) equation and is given by

$$\left[\beta^2 u - k \frac{u^2}{2} \right]_x + v_y = \left(\frac{S}{R} \right)_x \tag{1}$$

where u and v are perturbation velocities in the x and y directions, respectively, nondimensionalized with respect to freestream values. S is the entropy change, R is the gas constant and

$$\beta^2 = 1 - M_\infty^2; \quad k = (\gamma + 1)\, M_\infty^{1.5} \tag{2}$$

where M_∞ is the freestream Mach number. If ϕ is a perturbation velocity potential, and ψ is a vector potential, then

$$
\left.\begin{aligned}
u &= \phi_x + \psi_y \\
v &= \phi_y - \psi_x
\end{aligned}\right\} \tag{3}
$$

and

$$
\psi_{xx} + \psi_{yy} = -\omega \tag{4}
$$

where the vorticity ω is related to the entropy S by Crocco's theorem. The entropy generated [7] at a shock is given by

$$
\frac{S}{R} \simeq \frac{2\gamma}{(\gamma + 1)^2} \quad (M^{+2} - 1)^3 \tag{5}
$$

where M^+ is the Mach number just ahead of the shock and is related to u. The entropy generated by the shock is constant with respect to x aft of the shock. The tangency boundary condition is represented by the thin airfoil approximation so that the equation and its boundary conditions reduce to the classic TSD formulation in the absence of vorticity. The tangency boundary condition is given by

$$
(\phi_y - \psi_x)_{y = \pm o} = y_S'(x) \tag{6}
$$

where $y = y_s(x)$ denotes the airfoil.

Equation (1) with its boundary condition, Equation (6), and the vorticity equation, can be written in similarity form to give

$$\left(\bar{u} - \frac{\bar{u}^2}{2} \right)_x + \bar{v}_{\bar{y}} = \frac{-2\gamma}{(\gamma + 1)^2} k \beta^2 \frac{\partial}{\partial x} [(1 - \bar{u}^+)^3] \tag{7}$$

$$\bar{v}(x, \pm o) = k/\beta^3 \bar{y}_s'(x) \tag{8}$$

where

$$\bar{u} = \bar{\phi}_x + \beta^2 \bar{\psi}_{\bar{y}} \; ; \quad \bar{v} = \bar{\phi}_{\bar{y}} - \bar{\psi}_x \tag{9}$$

$$\beta^2 \bar{\psi}_{\bar{y}\bar{y}} + \bar{\psi}_{xx} = - \bar{\omega} = \frac{2\gamma}{(\gamma + 1)} \beta^3 \frac{\partial}{\partial \bar{y}} [(1 - \bar{u}^+)^3] \tag{10}$$

and $\bar{\phi} = {}^k/_{\beta^2}\, \phi$; $\bar{\psi} = {}^k/_{\beta^3}\, \psi$; $\bar{\omega} = {}^k/_{\beta^3}\, \omega$, $\bar{y} = \beta y$ and $\bar{u}^+$ is the value of $\bar{u}$ just ahead of the shock. In the derivation of Equation (10) a first order approximation to Crocco's Equation has been used; thus

$$\omega = \frac{1}{\gamma M_\infty^2} \frac{\partial}{\partial y} \; (S/R) \tag{11}$$

If $\tau \tilde{y}_s = y_s$ where τ is the thickness of the airfoil then a transonic similarity parameter, $\tilde{K}$, can be defined as

$$\tilde{K} = {}^{k\tau}/_{\beta^3} \tag{12}$$

The transonic similarity parameter, $\tilde{K}$, is constant for combinations of Mach number and thickness; if the airfoil is very thin the Mach number approaches unity and the terms due to entropy production vanish as $\tau \to o$ and $\beta^2 \to o$. Thus for a given $\tilde{K}$ it is possible to approach the TSD equation arbitrarily closely by varying the thickness and the Mach number.

The set of equations, Equations (7-10), can be solved using the same algorithm and treatment of the boundary conditions used in TSD example, thus ensuring that a phantom solution can be obtained as β and τ approach zero since in this case a potential solution is recovered.

If any vorticity is introduced into the two-dimensional flow the vorticity vector will be aligned with the spanwise direction and the magnitude of the vorticity will not change in this direction. According to the equations of Klopfer and Nixon (8) this vorticity is convected like

$$(\omega u)_x + (\omega v)_y = 0 \tag{13}$$

to a first approximation. The velocity induced by the vorticity is given by Equation (4) or Equation (10). In this model vorticity is defined as

$$v_x - u_y = \omega \tag{14}$$

Using Equation (14) in Equation (13) gives an alternative representation of the vorticity transport, namely,

$$A_{xx} + A_{yy} = \omega^2 + \omega_x v - \omega_y u \tag{15}$$

where

$$A_x = \omega\ v;\ A_y = -\ \omega\ u \tag{16}$$

A suitable boundary condition for Equation (15) is that normal derivatives of Λ are specified on the boundaries, that is, Neumann boundary conditions are used.

Equation (15) can be solved using a simple central differencing scheme with the vorticity terms on the right hand side set retarded by one iteration. This scheme is adequate for the small values of vorticity introduced in the present study Equation (15) can be solved iteratively with Equations (1), (3), (4), and (5).

In some of the examples discussed in the next section a small amount of vorticity, comparable in magnitude to the shock generated vorticity, is introduced over part of the chord. The amount of vorticity, ω, is symmetric on both surfaces of the airfoil.

The numerical method used is nearly the same for all cases and is based on the TSFOIL [9] code, which solved the TSD equation in similarity form. Further details can be found in Reference [6].

The TSD-Euler equation was used to predict the flow over a NACA 0012 airfoil at M_∞=0.85, α = 0 deg, and at M_∞ = 0.8 and α = 1 deg. The results are shown in Figures (2) and (3), respectively, and are compared to solutions of the TSD equation and the Euler equations. It may be seen that the nonlifting solution, Figure (2) compares quite well with the Euler solution. The lifting solution shown in Figure (3) is not in such good agreement. The probable reason is that the thin airfoil boundary conditions are not adequate. However, there is a considerable improvement over the TSD equation. It is interesting to note that the dominant effect of the entropy is the change in shock strength; computations which included only this effect gave identical results to those shown in Figures (2) and (3). This result confirms the practicability of using "nonisentropic" potential theories, such as that by Klopfer and Nixon [10], as an alternative to the Euler equations.

Phantom Solutions to the TSD-Euler Equation

Before discussing the occurrence of phantom solution to the TSD-Euler equation it is necessary to indicate how these solutions were found.

The form of the equation solved is based on the TSD formulation solved in the code TSFOIL; and is as follows

$$[K - (\gamma+1)\tilde{\phi}_{x} - (\gamma+1)\phi_{\tilde{y}}](\tilde{\phi}_{x} + \phi_{\tilde{y}}) + \tilde{\phi}_{\tilde{y}\tilde{y}} - M_\infty^{\tau^{-2/3}} \phi_{x\tilde{y}}$$

$$= \frac{2\gamma}{3(\gamma+1)^2} M_\infty^{2.75} \tau^{2/3} \{K - (\gamma+1)\tilde{\phi}_{x} - (\gamma-1)\phi_{\tilde{y}}\}^3 \tag{17}$$

where $\tilde{\phi}$ is related to the physical perturbation potential ϕ by

$$\tilde{\phi} = \tau^{-2/3} M_\infty^{3/4} \phi \tag{18}$$

and

$$\tilde{y} = \tau^{1/3} M_\infty^{1/2} y, \quad K = \frac{1-M_\infty^2}{\tau^{2/3} M_\infty} \tag{19}$$

The tangency boundary conditions for $\tilde{\phi}$ and ψ are

$$\tilde{\phi}_{\tilde{y}} = \frac{dy_s}{dx}, \quad \psi_{\tilde{y}} = 0 \tag{20}$$

where

$$y = \pm \tau \tilde{y}_s (x) \tag{21}$$

denotes the surface of the airfoil.

In the computations a factor ϵ is introduced such that K is kept constant. The factor ϵ is defined such that

$$M \tau^{2/3} = \epsilon M_o \tau_o^{2/3} \tag{22}$$

where M_∞ and τ are the new Mach number and thickness, respectively and M_o and τ_o are values of the Mach number and thickness for which phantom solutions exist for the TSD equation. Using Equation (22) and keeping K constant gives the following relations.

$$M_\infty = [1 - \epsilon (1 - M_o^2)]^{1/2} \tag{23a}$$

$$\tau = \tau_o [\epsilon M_o/M_\infty]^{3/2} \tag{23b}$$

In all of the computations discussed in this section M_∞, τ_o remain fixed for a given airfoil and ϵ is changed to increase the "Euler" effect. It may be noted that if ϵ is unity then M_o and τ_o are equal to the "potential" values M_∞ and τ. A value of ϵ that approaches zero means that the TSD-Euler equation approaches the classic TSD equation, thus ensuring that phantom solutions can be found in the limit.

The grid is completely symmetric and is the same as that used in earlier studies of the TSD equation. The boundary conditions at inflow are symmetric except when lift is present; in this case the code imposes a far field circulation to speed convergence. This is identical to that used in earlier studies [11] of the TSD equation and does not influence the appearance of multiple solutions.

The airfoil geometry for both sample airfoils is given in the code by an analytic formula and symmetry is imposed directly.

Vorticity is introduced through the surface on several of the examples. The vorticity transport is modeled by Equation (13) and the vorticity flux at the boundary is added between about 10% and 45% of chord according to the distribution

$$\omega = \omega_o \{ 1 + \cos [(\pi (i-i_s)/|i-i_s|)]\} F \tag{24}$$

where i_s is a specified point (in most cases at $x = 0.265$) and i is the number of grid points over which the vorticity is introduced; i_s is the mean value of the indices between which the vorticity is introduced. The factor F is introduced to provide a skewness to the distribution in an attempt to model the physical flow better and is defined by

$$\left.\begin{array}{ll} F = 0 & i < i_s \\ F = \exp\left(-|i-i_s|\right) & i > i_s \end{array}\right\} \tag{25}$$

A negative value of ω_o indicates an entropy source.

Results

Two generic airfoil sections were used in the study, namely a NACA 00XX section and a biconvex section. The parameter K given by Equation (19) for the NACA 00XX section is kept constant at the value for $M_\infty = 0.85$ and $\tau = 0.12$, a condition for which phantom solutions to the TSD equation can be found. For the biconvex section K is kept constant at the value for $M_\infty = 0.81$ and $\tau = 0.1$. In the following descriptions of the computations the parameter ϵ is used to differentiate the results; the corresponding values of M_∞ and τ for the NACA 00XX and biconvex sections can be found in Table I and Table II, respectively. The phantom solutions are initiated by giving the airfoil an angle of attack for some iterations before putting the angle of attack to zero.

In an earlier paper it was noted that phantom solutions could not be found for this airfoil unless the Mach number was close to unity (actually $\epsilon = 0.1$) but by the introduction of vorticity, ω_o, the Mach number range of the occurrence of phantom solutions could be improved. In the present study values of ϵ between zero and 0.6 were used and it was found that from a conventional start, that is, the computation is started as described earlier from freestream, phantom solutions could not be found for $\epsilon > 0.2$ even when vorticity is introduced; for $\epsilon = 0.2$ the value of ω_o is -0.125. However, if the starting sequence was altered such that the initial flow is not freestream, then it proved possible to obtain phantom solutions for $\epsilon = 0.4$ with $-0.105 < \omega_o < -0.145$. In these results the lift coefficient was approximately constant at a value of 0.33. Examples of phantom solutions for $\epsilon = 0.2$ and $\epsilon = 0.4$ are shown in Figures (4) and (5), respectively.

The fact that the occurrence of phantom solutions depends on the starting solutions indicates that there may be more such solutions than would commonly be estimated using a conventional start to the computation.

A phantom solution for the biconvex airfoil is shown in Figure (6); the value of ϵ is 0.04 and $\omega_o = -0.01$. In general it proved more difficult to obtain phantom solutions for the biconvex airfoil.

The introduction of vorticity through the boundary has two effects on the computation, namely it introduces entropy into the flow field and it induces a velocity on the airfoil surface. The induced velocity on the airfoil surface in a thin airfoil formulation is equivalent to changing the airfoil shape. Consequently it is of interest to determine if the dominant effect of the vorticity introduction in the appearance of phantom solutions is the change in the effective airfoil geometry or the entropy produced in the flow field. Therefore, the computer code was run with the change in geometry being only effect of the vorticity introduced through the surface. It is found that this code gave essentially the same result as the original code indicating that the dominant effect of the vorticity introduction is the change the airfoil geometry and suggesting that airfoils could be designed to give phantom solutions.

Test for Phantom Solutions

The TSD-Euler equation [5] can be written in the form

$$\phi_{xx} + \phi_{yy} = - \frac{\partial}{\partial y} \bullet \{M_\infty^2 u + \frac{ku^2}{2} + (1 + \frac{1}{\gamma M_\infty^2}) \ S/R\} \tag{26}$$

where

$$\phi_y = \beta^2 u - \frac{ku^2}{2} \ ; \ \phi_x = -v \tag{27}$$

and β^2 and k are given by Equation (2).

If the substitution

$$\bar{y} = \beta y \tag{28}$$

is introduced then Equation (24) becomes

$$\bar{\psi}_{xx} + \bar{\psi}_{\bar{y}\bar{y}} = -\beta \frac{\partial}{\partial \bar{y}} \left\{\frac{k}{\beta^2} \frac{u^2}{2} + \frac{1}{\beta^2} \left(1 + \frac{\beta^2}{\gamma M_\infty^2}\right) S/R\right\} \tag{29}$$

Using an analysis similar to that in Reference (5), Equation (27) can be written in integral form over the flow field D to give

$$\tilde{u} - \frac{\tilde{u}^2}{2} - \bar{S} = \int_o^1 \{ \Delta_o \bar{\psi}_\xi K_{o_x} + \Delta_o \bar{\psi}_\eta K_{o\bar{y}} \} d\xi - \int_o^\infty \Delta_o \left[\frac{u^2}{2} + \left(1 + \frac{\beta^2}{\gamma M_\infty^2}\right)\Delta_o \tilde{S}\right] d\xi$$

$$- \iint_D K_{\eta\bar{y}} \left\{ \frac{\tilde{u}^2}{2} + \tilde{S} \left(1 + \frac{\beta^2}{\gamma M_\infty^2}\right)\right\} d\xi d\eta \tag{30}$$

where

$$\tilde{u} = \frac{k}{\beta^2} u; \quad \tilde{S} = \frac{k}{\beta^4} \frac{S}{\kappa} \tag{31}$$

$$\left.\begin{aligned} K &= \frac{1}{2\pi} \ln|(x-\xi)^2 + (\bar{y}-\eta)^2| \\ K_o &= \frac{1}{2\pi} \ln|(x-\xi)^2 + \bar{y}^2| \end{aligned}\right\} \tag{32}$$

and

$$\left.\begin{aligned} \Delta_o f &= \frac{1}{2} [f\ (x,+o) - f\ (x,-o)] \\ \Delta f &= \frac{1}{2} [f\ (x,\bar{y}) - f\ (x, -\bar{y})] \end{aligned}\right\} \tag{33}$$

Equation (28) is valid for $\bar{y} \neq 0$: on $\bar{y} = o$ only the symmetric part of Equation (28) exists and the asymmetric component is given by

$$- \bar{v}\ (x, \pm o)\Big|_{average} = \int_o^1 K_{ox}\ \Delta_o\ \bar{\psi}_\eta d\xi - \iint_D K_{x\eta} \left[\frac{\bar{u}^2}{2} + \bar{S}\ \left(1 + \frac{\beta^2}{\gamma M_\infty^2}\right)\right]\ d\xi d\eta \tag{34}$$

For the case of phantom solutions when the airfoil is symmetric and at zero angle of attack. $\bar{v}|_{average}$ is zero and Equations (28) and (32) can be arranged to give equations for $\Delta\tilde{u}$; thus

$$\Delta\tilde{u}\ (1-\bar{u}) = - \iint_{D_1} K_{\eta\bar{y}}\ [\Delta\widetilde{u\bar{u}} - \Delta_o\tilde{u} + \left(1 + \frac{\beta^2}{\gamma M_\infty^2}\right)\ \Delta\tilde{S} - \frac{\beta^2}{\gamma M_\infty^2}\ \Delta_o\tilde{S}]\ d\xi d\eta \tag{35}$$

$$\iint_{D_1} K_{ox\eta}\ [\ \Delta\widetilde{u\bar{u}} - \Delta_o\tilde{u} + \left(1 + \frac{\beta^2}{\gamma M_\infty^2}\right)\ \Delta\tilde{S} - \frac{\beta^2}{\gamma M_\infty^2}\ \Delta_o\tilde{S}]\ d\xi d\eta = 0 \tag{36}$$

where a barred quantity is defined by

$$\bar{f} = \frac{1}{2}\ [f\ (x,y)\ +\ f\ (x,-y)] \tag{37}$$

and D_1 is the upper half plane.

$\tilde{S}$ is related to $\tilde{u}$ through Equation (5) and Equation (29). Thus Equations (33) and (34) are equations for $\Delta\tilde{u}$ in terms of the average value of $\tilde{u}$, $\bar{u}$. The integrals can be discretized by a sum; thus the integral in Equation (33) can be represented by

$$- \iint_{D_1} K_{\eta \bar{y}} \, g \, d\xi d\eta \simeq \tilde{A}_{ij} g_j ; \qquad i = N_x, \ N_T \tag{38}$$

while the integral in Equation (34) can be represented by

$$\iint_{D_1} K_{x\eta} \, g d\xi d\eta = \tilde{B}_{ij} \, g_j \qquad i = 1, \ N_x \tag{39}$$

where N_x and N_y and the number of grid points in the x and y directions respectively and

$$N_T = N_x N_y \tag{40}$$

Since $\Delta\tilde{S}$ is a function of $\Delta\tilde{u}$ and $\bar{u}$ Equations (33) and (34) can be represented by the following matrix equation.

$$A_{ij} \, \Delta\tilde{u}_j = 0 \qquad i = 1 \ N_T \tag{41}$$

Note that both Equation (33) and Equation (34) have been combined to give this result. Note also that A_{ij} is a function of $\bar{u}$ and $\Delta\tilde{u}$. For the potential TSD formulation A_{ij} is a function only of $\bar{u}$.

Suppose that a phantom solution can occur with an infinitesimal amount of asymmetry that is $\Delta\tilde{u}$ can be vanishingly small; evidence for $\Delta\tilde{u}$ being vanishingly small can be inferred from the results of Williams et al(4) where the phantom solution can have infinitesimal lift. In such a case the symmetric solution bifurcates at some point and at this bifurcation point both symmetric and asymmetric solutions are possible. This can only occur if the

$$\det A_{ij} = 0 \tag{42}$$

where "det ()" denotes the determinant of the matrix A_{ij} at this point. Since at this point $\Delta\tilde{u}$ is vanishingly small A_{ij} is a function of $\bar{u}$ and hence Equation (40) is an equation for $\bar{u}$. Since in the TSD approximation $\bar{u}$ is related to the local Mach number distribution (M) by

$$\bar{u} = \frac{M^2 - M_\infty^2}{1 - M_\infty^2} \tag{43}$$

Equation (40) is an equation for the Mach number distribution in terms of the freestream Mach number. Hence, the occurrence of phantom solutions is dependant only on the Mach number distribution for a fixed M_∞. In other words there is a Mach number distribution for which the solution can bifurcate, indicating that phantom solutions can be "designed" to appear. This fact is borne out by the results of the vorticity introduction study which indicated that a dominant cause of phantom solutions is the airfoil geometry. Thus, it should be possible to design an airfoil which will have phantom solutions for a given M_∞.

In order to test whether the phantom solution satisfy Equation (36) the phantom solution for the Joukowski airfoil in potential flow at $M_\infty = 0.85$ can be used to evaluate the integral. However, evaluation of the integral is not satisfactory, principally due to the fact that the shock is captured and thus is not a solution of the differential equation, furthermore, at the shock the algorithm has a truncation error of $O(1)$. A better means of judging whether the integral is zero is to used an inverted form of Equation (36), namely,

$$F(x) = \frac{1}{2}\left[\Delta u(x) - \frac{\Delta u^2}{2}(x)\right] + \frac{1}{\pi}\left(\frac{1-x}{x}\right)^{1/2}\int_0^1 \frac{I_c(\xi)}{x-\xi}\left(\frac{\xi}{1-\xi}\right)^{1/2} d\xi = 0 \tag{44}$$

where, for the TSD equation,

$$I_c(\xi) = \iint_{D_1} K_{ox\eta}\,[\,\Delta\tilde{\bar{u}}u - \Delta_o u\tilde{\bar{u}}_o]\; d\xi d\eta \tag{45}$$

In this formulation the integral $I_c(\xi)$ is not so sensitive to errors since on $\eta = 0$ the integrand is zero and the increasing error as $\eta \rightarrow \infty$ is counteracted by the decreasing of the kernel function $K_{ox\eta}$ as $y \rightarrow \infty$. Since for a normal shock the first term is a smooth function the error in the shock capture region can be reduced by interpolation of this term for points inside the shock capture region. As noted earlier, the integral I_c, and its inverse, should not be too inaccurate. The variation of F(x) and x is shown in figure 7. The value for the Joukowski airfoil at $M_\infty = 0.81$ and $\alpha = 1/4°$ and its exact, theoretical solution, are shown for comparison. It can be seen that the F(x) is very close to zero (apart from the shock region) and strongly suggests that the phantom solutions satisfy Equation (36). Evaluations of the integral for other configurations lead to the same conclusion. Thus, while it is unlikely that the finite difference results will give the singular matrix, det A_{ij}, at the same freestream conditions as the actual occurrence in the calculations. The determinant should become singular at approximately the correct Mach number.

In order to test the criterion of Equation (40) further the determinant of A_{ij} was evaluated by performing a LU decomposition of the matrix and identifying the smallest magnitude of the elements on the diagonal. For a singular matrix this magnitude should be zero. As a test a true singular equation, based on Equations (33) and (34) was set up by multiplying the integrals be a factor such that these equations are satisfied identically by a finite difference phantom solution. The smallest element on the diagonal of the LU decomposition is then taken to be zero in subsequent cases. Note that the smallest element is not zero because of machine error and the overall magnitude of the determinant; in the present study the determinant itself could be as small as exp(-200).

For the TSD equation the calibration test noted earlier gives a minimum magnitude of 5.64×10^{-4} and in the subsequent tests anything less than this value is assumed to be (probably) zero. The machine used is a VAX 11/750 computer. In Figure (8) the magnitude of the minimum diagonal element is shown for various values of M_∞ for NACA 0012, 14% biconvex and 11% Joukowski airfoil. It can be seen that phantom solutions can be found in the neighborhood of $M_\infty \sim 0.81$ for the NACA 0012, $M_\infty \sim 0.88$ for the Joukowski airfoil and $M_\infty \sim 0.86$ for the biconvex airfoil, $M_\infty \sim 0.86$. The range for multiple solutions for the NACA 0012 and 11.8% Joukowski airfoils using the TSD equations is 0.82 - 0.86, and for the 14% biconvex airfoil, 0.79-0.83.

Although the test appears to work there are some aspects of the derivation of the test that deserve a warning. First, the basis of the test is that there is a bifurcation point where the symmetric solution can change smoothly to a phantom solution. It is not proven that such a bifurcation point exists, although the results of Reference [4] do indicate such a bifurcation. Secondly, the inability of the present formulation to give machine zero (10^{-6}) for a real singular equation can cast doubt on the conclusions.

The above tests are for the TSD equation. A similar series of tests were computed for the TSD-Euler equation. In these tests the parameter ϵ in Equation (23) is picked so that the airfoil thickness is constant for a range of Mach numbers. The thickness is picked so that at $M_\infty = 0.986027$ there is a phantom solution. Results of the test are shown in Figure (9). The "zero" obtained in a similar fashion as for the TSD equation for a grid similar to that used in the TSD-Euler calculations is 2.1×10^{-3}. It may be seen from Figure (9) that the test does not work as well for the TSD-Euler equation as for the TSD-Equation although it is possible that there is a bifurcation at $M_\infty = 0.9753$ which gives a value close to the "zero" value.

The test described above is logical in a mathematical sense and can give insight into the occurrence of phantom solutions. However, as a predictor or a design criteria for phantom solutions it requires further work to remove the obstacles of numerical error accumulation and the fact that the numerical solutions do not satisfy the integral equations that are the basis of the theory.

Multiple Solutions of the Navier-Stokes Equations

The type of phantom solutions discussed in the previous sections have been solutions of the TSD or TSD-Euler equation. However, it is known that multiple solutions of the Navier-Stokes equations can occur at transonic speeds. For example the results of Levy [12] showed that a 18% biconvex airfoil at zero angle of attack will give an oscillatory solution for a narrow range of Mach numbers. Other solutions of this type have been discussed by Mabey(13) and a range of thickness parameters for which these multiple solutions appear has been determined. The occurrence of these solutions are not limited to the biconvex airfoil. The term multiple solutions is used here because both the symmetric and the oscillatory solutions have been observed in wind tunnel tests so the terminology "phantom" solution may be misleading. In the present study the computer code ARC2D [14] was used to computer these oscillatory solutions for a 14% biconvex airfoil at zero angle of attack and $M_\infty = 0.81$; the Reynolds number is 6.9×10^6. The trace of the lift coefficient with time is shown in Figure (10).

The range of Mach numbers in which these oscillatory solutions appear is close to that for which phantom solutions appear in the TSD equation and it is a natural question as to whether the oscillatory solution for the Navier-Stokes equations in initiated by a mechanism similar to that for the inviscid flows. The oscillation is probably sustained by viscous interaction. Accordingly a series of calculations were performed to try to stabilize the asymmetric solution. This was attempted by a symmetric boundary layer suction and by attempting a solution for a very thin airfoil at M_∞ close to unity to mimic the results for the TSD-Euler equation. The suction failed to produce a stable solution while the very thin airfoil flow became dominated by the boundary layer and attempts to remove this by suction were not successful.

Although the attempts to produce a stable solution for the Navier-Stokes equations failed it is necessary to note that the Navier-Stokes equations do have multiple solutions in a small Mach number range since there is one symmetric and one oscillatory solution.

If the Navier-Stokes equations are regarded in the same light as an inviscid flow with a boundary layer it is possible to construct a scenario for the appearance of the multiple solutions. If the boundary layer can be represented by a displacement thickness then the symmetrical flow at the start of the oscillatory cycle can be regarded as an inviscid flow over a modified airfoil section. Alternatively the flow outside a boundary which borders the boundary layer can be regarded as inviscid. Since it has been shown that the effect of entropy as regards the appearance of phantom solutions is mainly to modify the Mach number distribution this inviscid, nonisentropic flow can give phantom solutions. Suppose now that at the start of the oscillatory cycle a phantom solution, in the inviscid sense, occurs. This solution will create circulation and drive one shock wave aft and the other forward. Since the shock movement will be associated with a change in shock strength the boundary layer separation region will change in both extent and strength causing a change in displacement thickness which changes the effective shape of the airfoil. This modified airfoil could be outside the range of phantom solutions so the solution will then tend towards symmetry. Since this is a dynamic process an oscillatory cycle could be produced. This scenario is really only speculation but some evidence for its existence can be provided by a simple time dependent computation.

Consider the potential flow over a symmetric airfoil at zero angle of attack in the range of phantom solutions. A small perturbation to the boundary is given so that a phantom solution starts to be generated. As this happens one shock will get stronger and a viscous flow would separate the boundary layer. The added displacement thickness from this separation can be represented, at least approximately, by the "viscous wedge" model suggested by Rizzetta & Yoshihara [15]. In the present formulation the airfoil slope is changed by the addition of the term $\frac{d\delta}{dx}*$ where

$$\frac{d\delta^*}{dx} = \begin{cases} 40\,\theta\,(x-x_s); & x_s \le x < x_s + 0.025 \\ \theta; & x_s + 0.025 \le x < x_s + 0.055 \\ 100\,\theta\,(x-x_s-0.155); & x_s + 0.055 \le x < x_s + 0.155 \\ \theta; & x_s + 0.155 < x \end{cases} \tag{46}$$

where x_s is the shock location and θ represents the slope of the separation displacement thickness, given by

$$\theta = \begin{cases} 0, & M^+ < 1.3 \\ \theta_o\,(M^+ - 1), & M^+ > 1.3 \end{cases} \tag{47}$$

M_+ is the Mach number just ahead of the shock; θ_o is to be determined.

The computer code XTRAN2L [14] was modified to incorporate the viscous wedge; XTRAN2L solves the time accurate TSD equation. The airfoil used in the test was a 14% biconvex airfoil at $M_\infty = 0.81$, the same as in the Navier-Stokes computations. The phantom solution was obtained by an initial perturbation and the shock strength on the upper surface was such that $M^+ > 1.3$ thus triggering the viscous wedge. Several values of θ_o were tried and the response of $\frac{d\delta}{dx}^*$ to the shock was lagged in time. In order to reproduce qualitatively the Navier-Stokes solution, θ_o was taken to be 0.32, considerably larger than that recommended by Rizzetta and Yoshihara [14] ($\theta_o = 0.09$); the viscous wedge was introduced 50 time steps behind the shock motion. In the subsequent oscillation this is equivalent to a lag of approximately a quarter of a cycle. The oscillation of the lift coefficient is shown in Figure (11). Only part of the cycle is shown; the cycle persists for the complete calculations, consisting of 1500 time steps. It can be inferred from these computations that a plausible initial cause of the oscillatory solutions of the Navier-Stokes equations is the same as that for the inviscid cases but that viscous forces are responsible for the oscillatory behavior. It is important to remember that the Navier-Stokes solutions do occur in wind tunnel tests and are not a mathematical or computational anomaly.

Concluding Remarks

The TSD-Euler equation, derived in previous work was examined for phantom solutions in an attempt to determine the effect on entropy and vorticity on the appearance of such solutions. It was found that phantom solutions can be found for this equation although for a different range of Mach numbers than for the TSD equation, at least for the airfoils considered. Following earlier work a source of vorticity was introduced on the airfoil surface and this enhanced the probability of the appearance of phantom solutions. Closer study indicated that the dominant effect of the vorticity introduction is to change the effective geometry of the airfoil with the implication that an airfoil could be designed to give a phantom solution to the TSD-Euler equation. The vorticity introduction can be regarded as a method of flow control which mimics the effect of a change in geometry.

A test of when a phantom solution can appear has been developed and it indicates that the dominant feature determining the appearance of phantom solutions is the local Mach number distribution; the entropy or vorticity in the flow field plays only a secondary role. The numerical verification of the test has some problems due principally to the numerical inaccuracies in the numerical solutions of the flow and in the evaluation of the test parameters. The test indicates that a phantom solution can be initiated by designing the airfoil to give a suitable distribution of Mach number in the flow field.

The inference of the study of the TSD-Euler equation is that entropy and vorticity do not inhibit the appearance of phantom solutions but simply change the freestream conditions for which these solutions can appear. Since vorticity does not inhibit the appearance of the phantom solutions it is pertinent to ask if the Navier-Stokes equations have phantom solutions. It is known that for a small Mach number range that the Navier-Stokes equations do have multiple solutions, are symmetric and are oscillatory. This behavior was shown to be explainable by the idea of phantom solutions, although this may not be the only explanation. The multiple solutions of the Navier-Stokes equations are apparent in wind tunnel tests, indicating their realism. The question that now arises is whether phantom solutions can be made to exist stably in a real environment thus introducing a new area of study for aerodynamicists and, possibly, introducing a new mechanism for flight. In order to achieve this the phantom solutions must be stable and some form of flow control is necessary.

If the test derived in this present work can be made more accurate numerically it should be possible to design an airfoil to give phantom solutions over a range of flight conditions. Again, some form of flow control is necessary. If such an airfoil can be designed then phantom solutions to the Navier-Stokes equations may be found, together with some means of stabilizing such solutions. After this stage a wind tunnel test could be conducted. To achieve such an end it is necessary to learn much more about the nature of all phantom solutions, not just those considered here, that is, the transonic solutions.

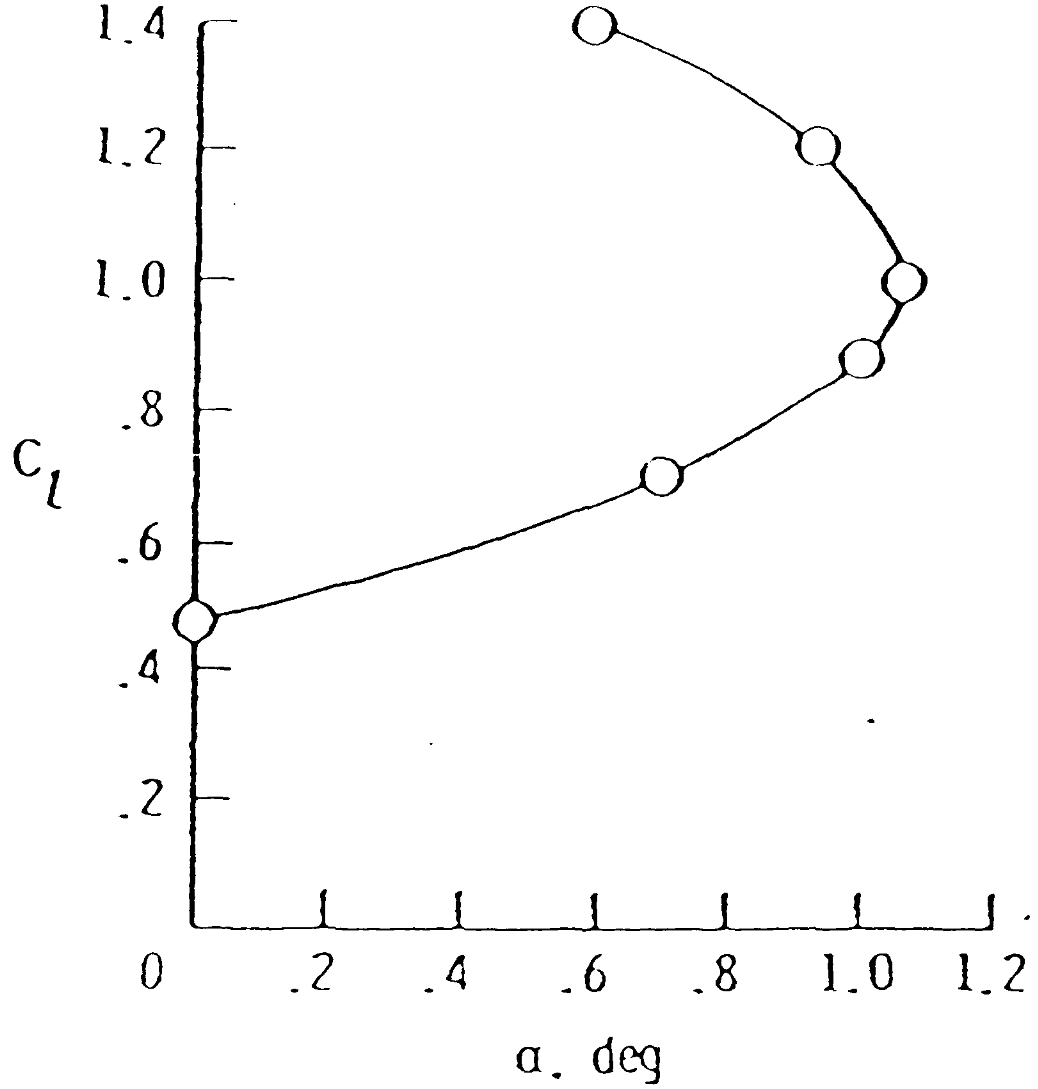

Figure 1.- C_L-α curve obtained with FLO 36 for an RAE 2822 airfoil.

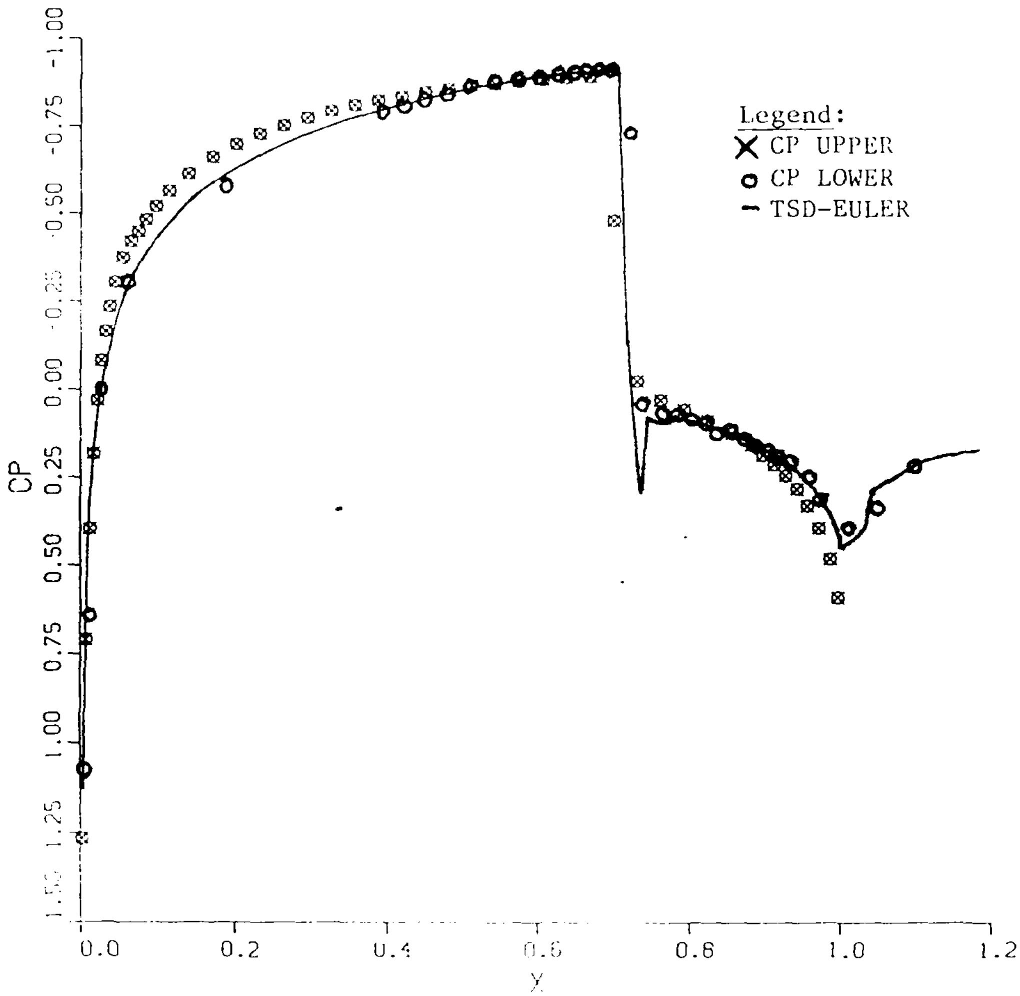

Figure 2.- Pressure distribution around a NACA 0012 airfoil; $M_\infty = 0.85$, $\alpha = 0.0$ deg.

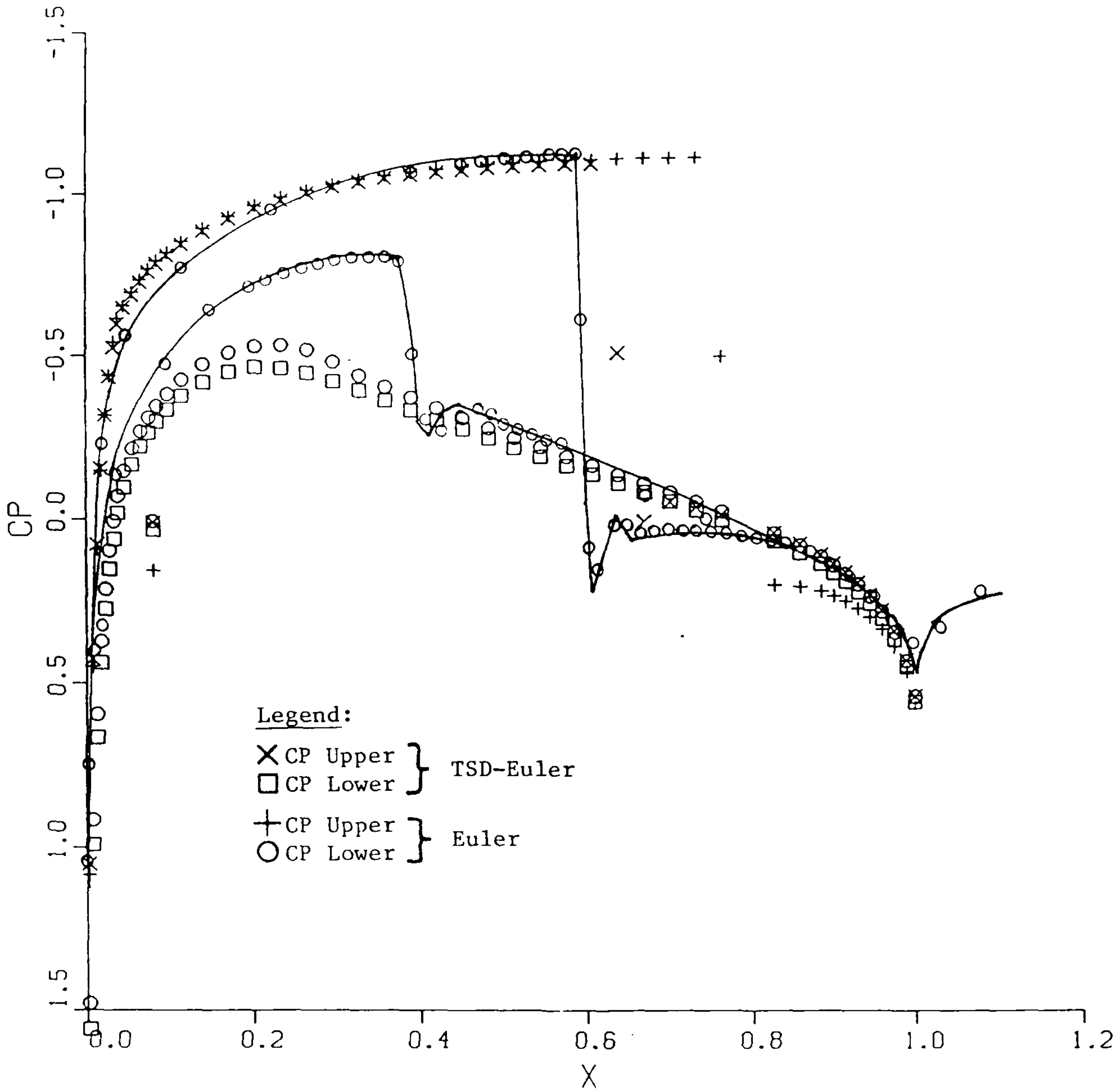

Figure 3.- Pressure distribution around a NACA 0012 airfoil, $M_\infty = 0.80$, $\alpha = 1.0$ deg.

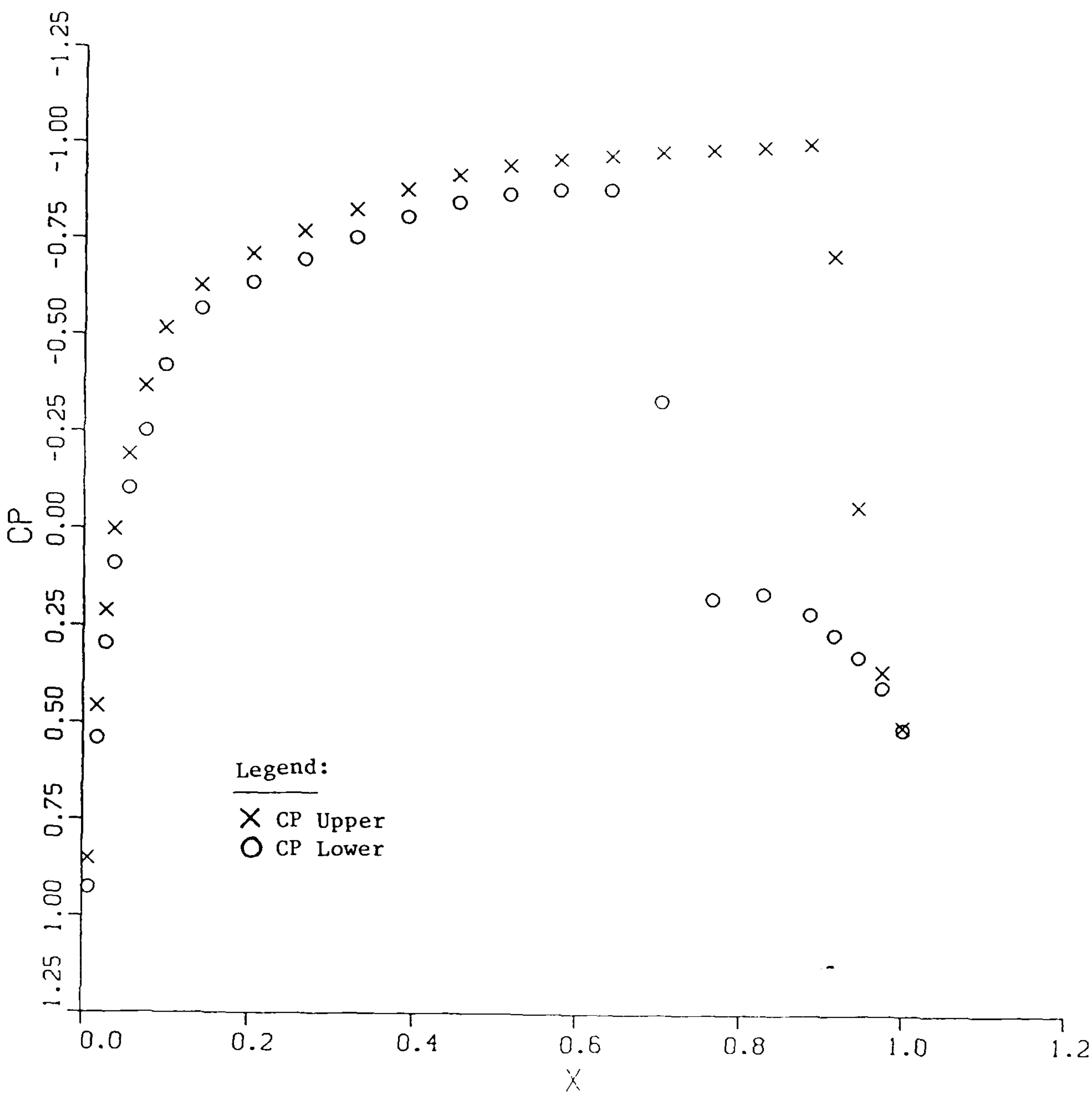

Figure 4.- Pressure distribution around a NACA 00XX airfoil; using the TSD-Euler equation; $M = 0.85$, $\varepsilon = 0.2$, $\Omega_0 = -0.125$.

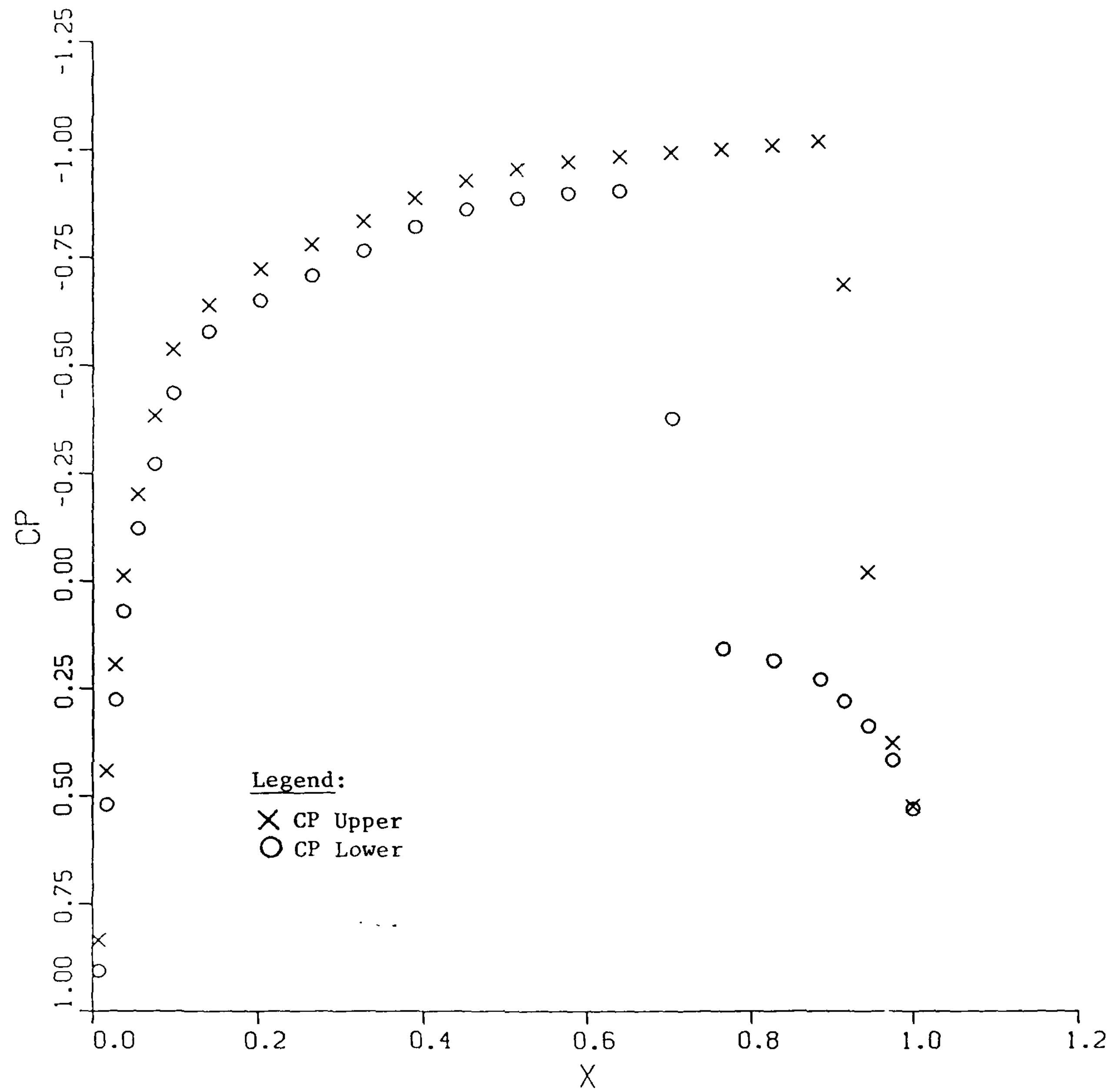

Figure 5.- Pressure distribution around a NACA 00XX airfoil; using the TSD-Euler equation; $M = 0.85$, $\varepsilon = 0.4$, $\Omega_o = -0.135$.

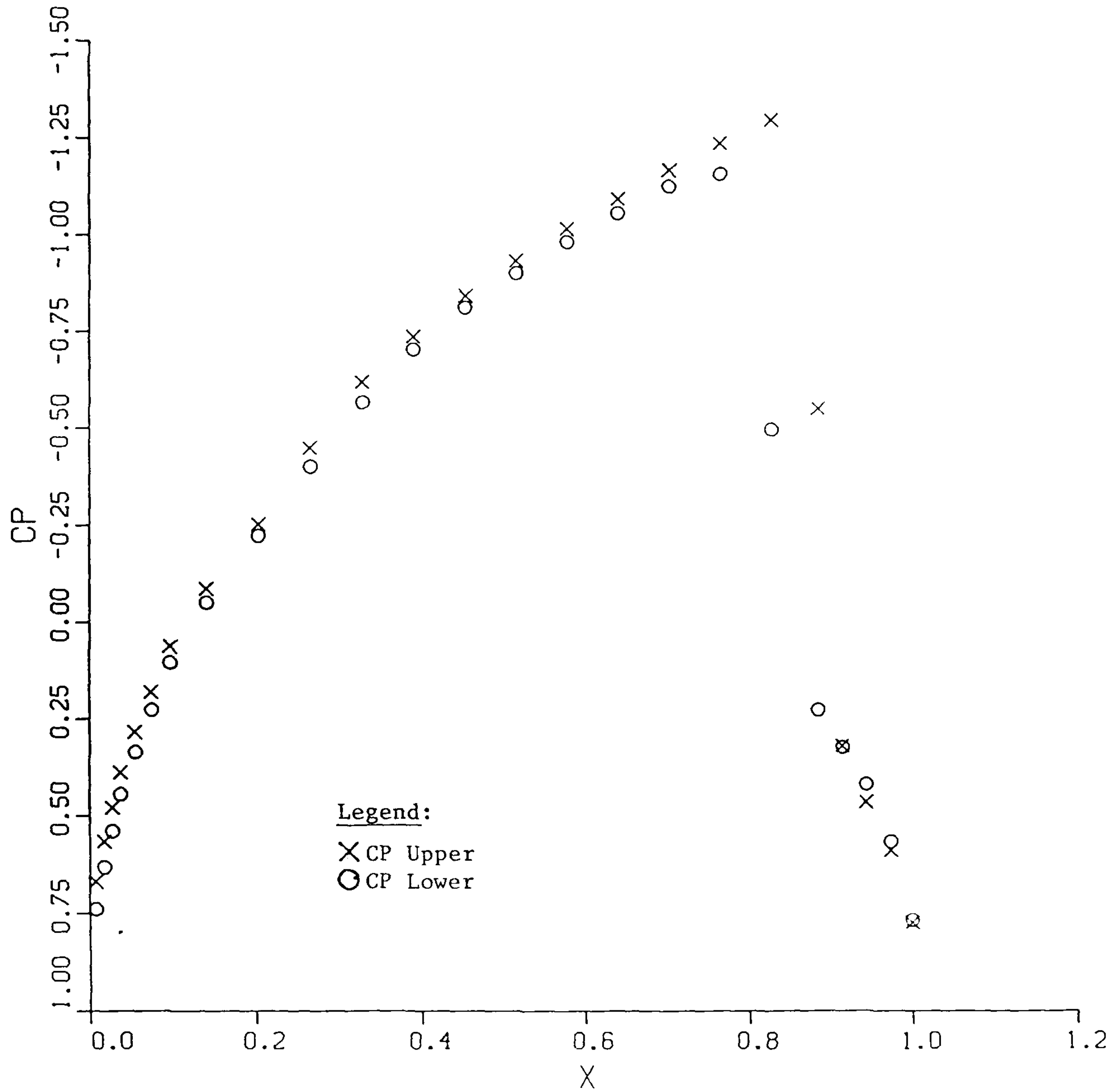

Figure 6.- Pressure distribution around a [illegible] airfoil using the TSD-Euler equation; $M = 0.81$, $\varepsilon = 0.04$, $\Omega_0 = -0.01$.

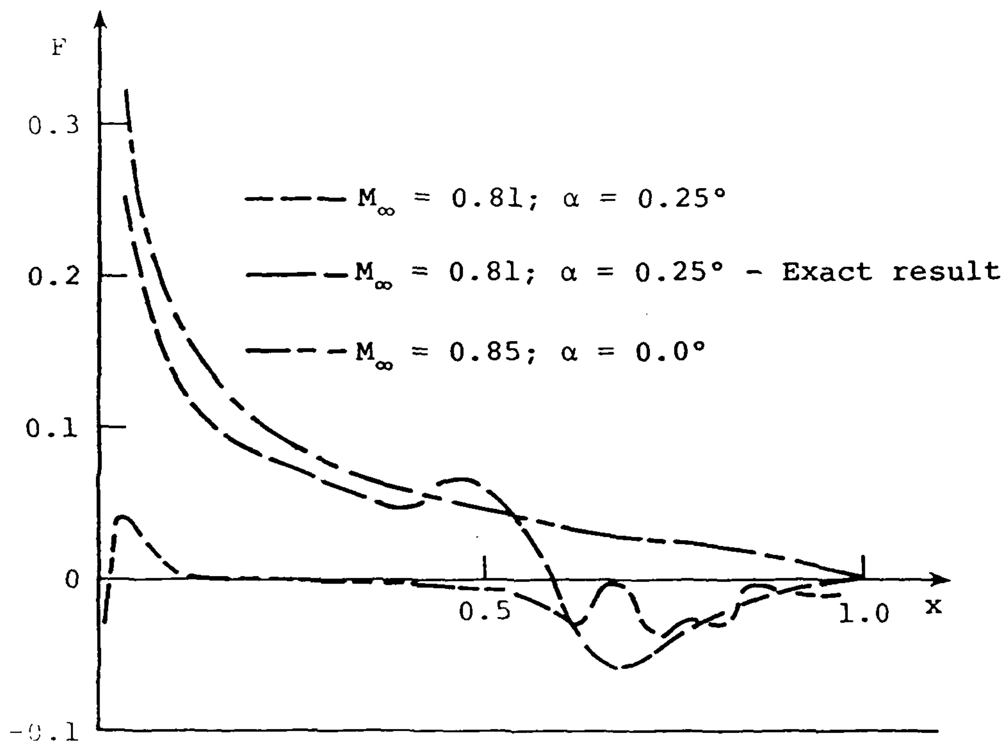

Figure 7.- Variation of F (x) with x.

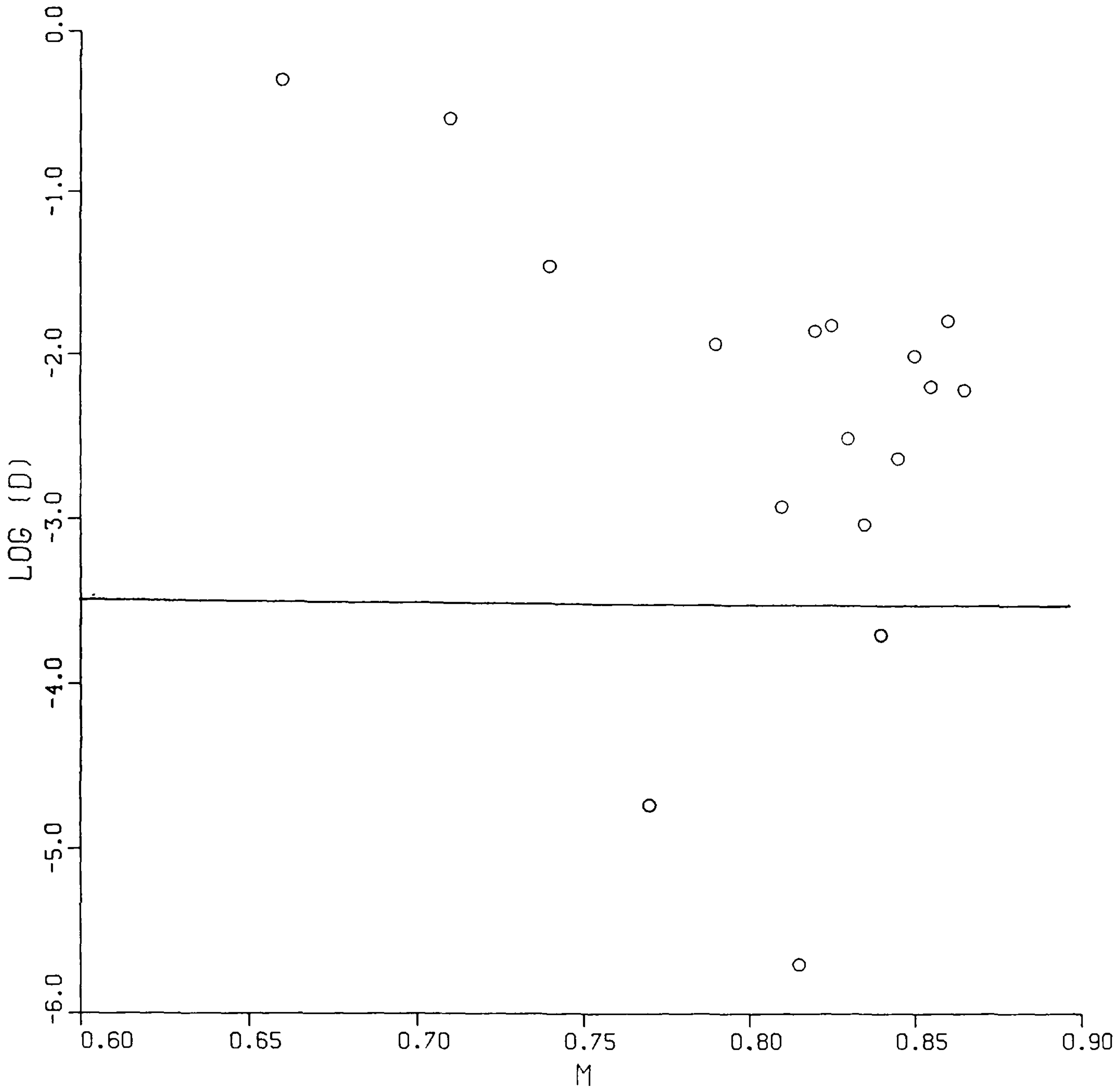

Figure 8a.- Test for phantom solutions for TSD equation; NACA 0012 airfoil.

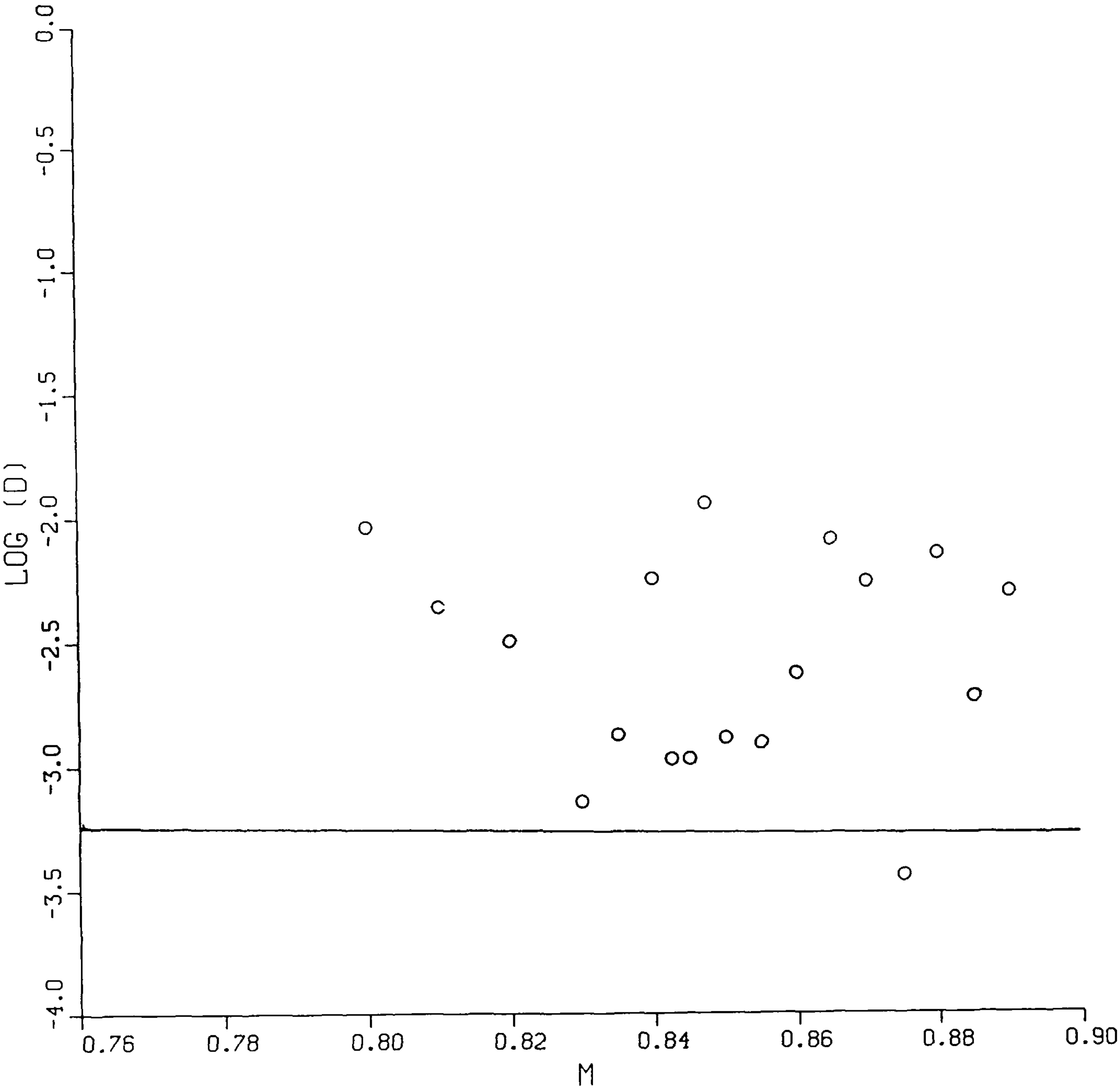

Figure 8b.- Test for phantom solutions for TSD equation; 12% biconvex airfoil.

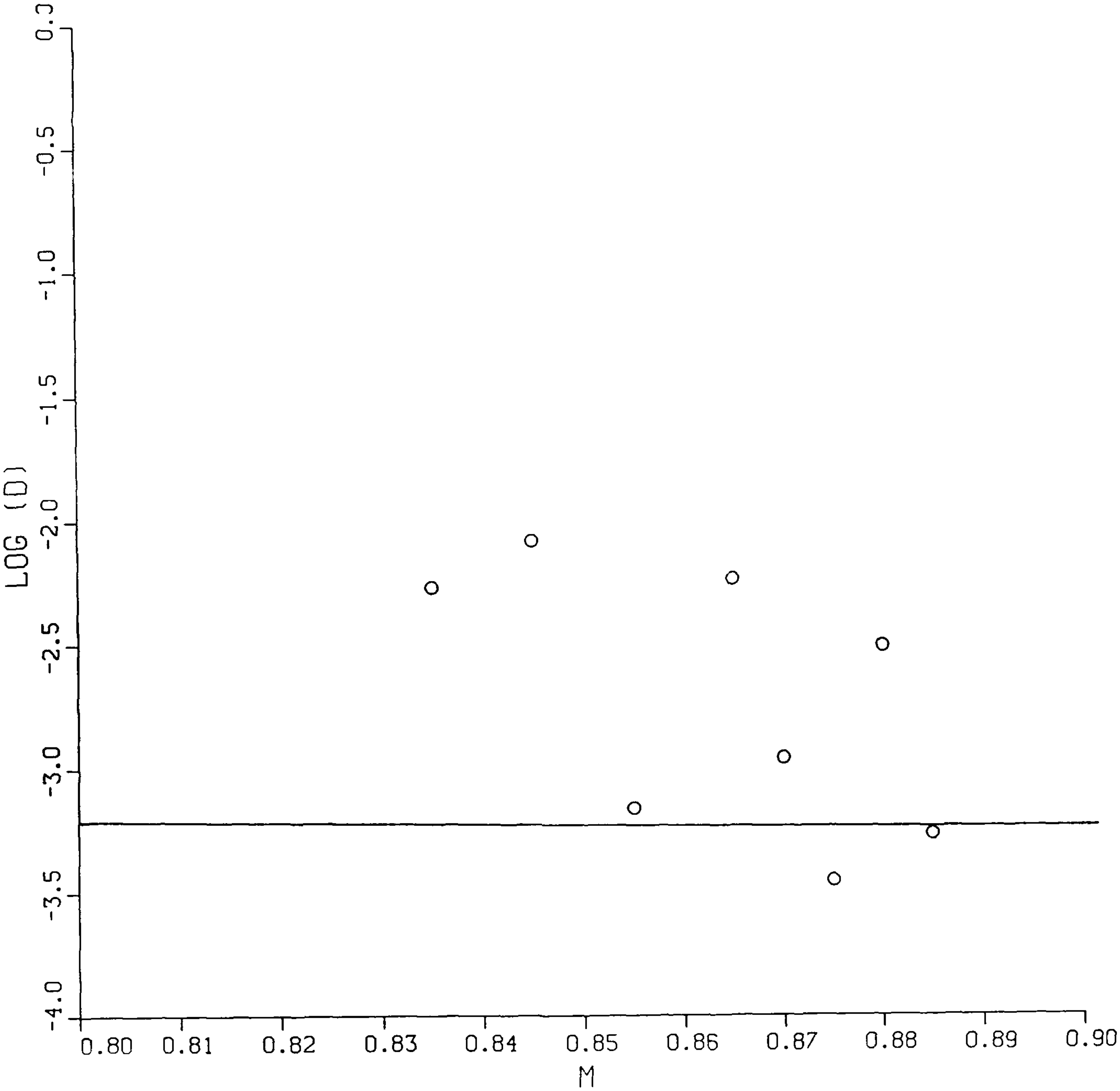

Figure 8c.- Test for phantom solutions for TSD equation; 11% Joukowski airfoil.

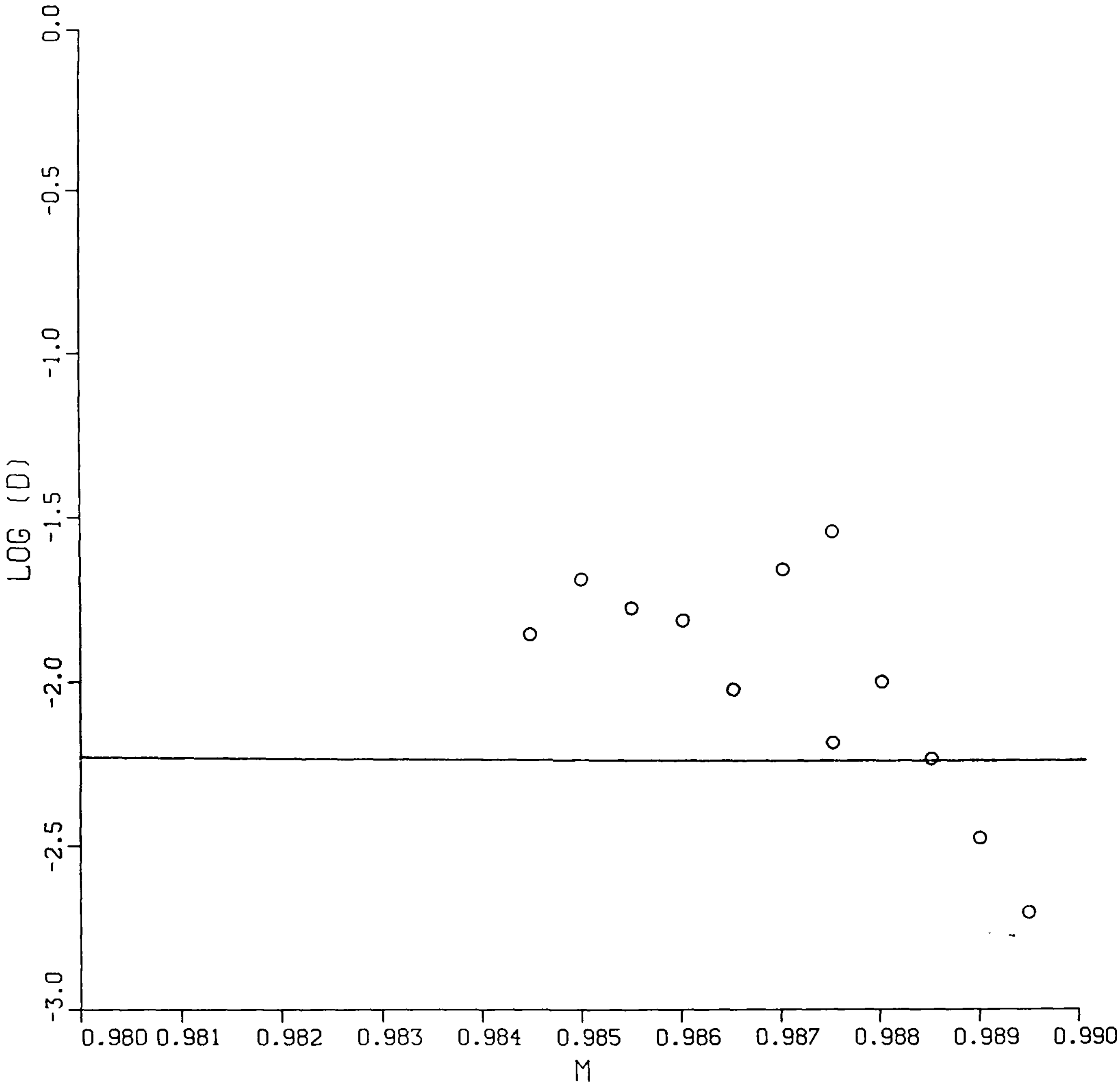

Figure 9.- Test for phantom solutions for TSD-Euler equation; NACA 00XX airfoil.

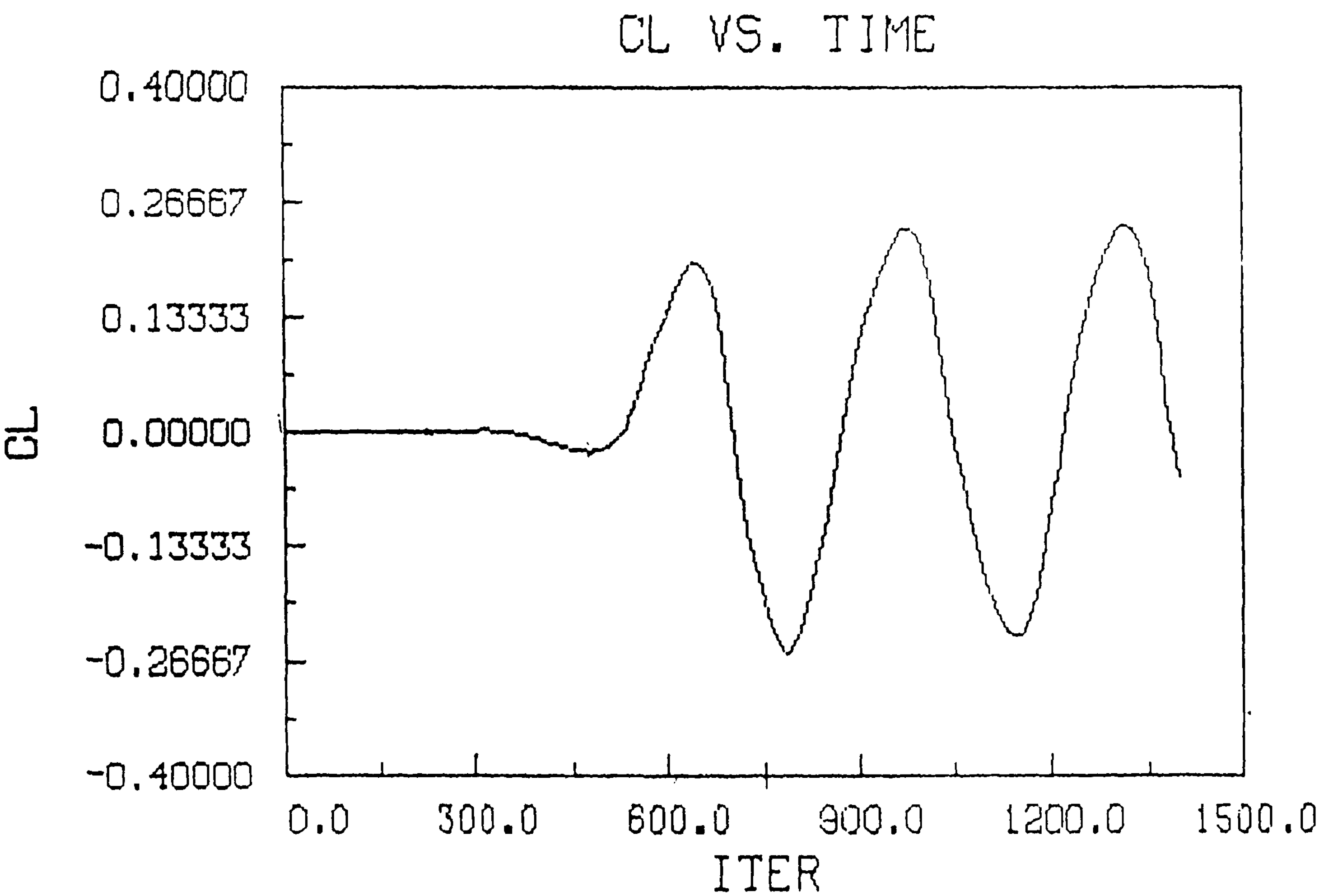

Figure 10.- Trace of C_L vs. time for 14% biconvex airfoil, M = 0.81, α = 0.0 deg; Navier-Stokes solution.

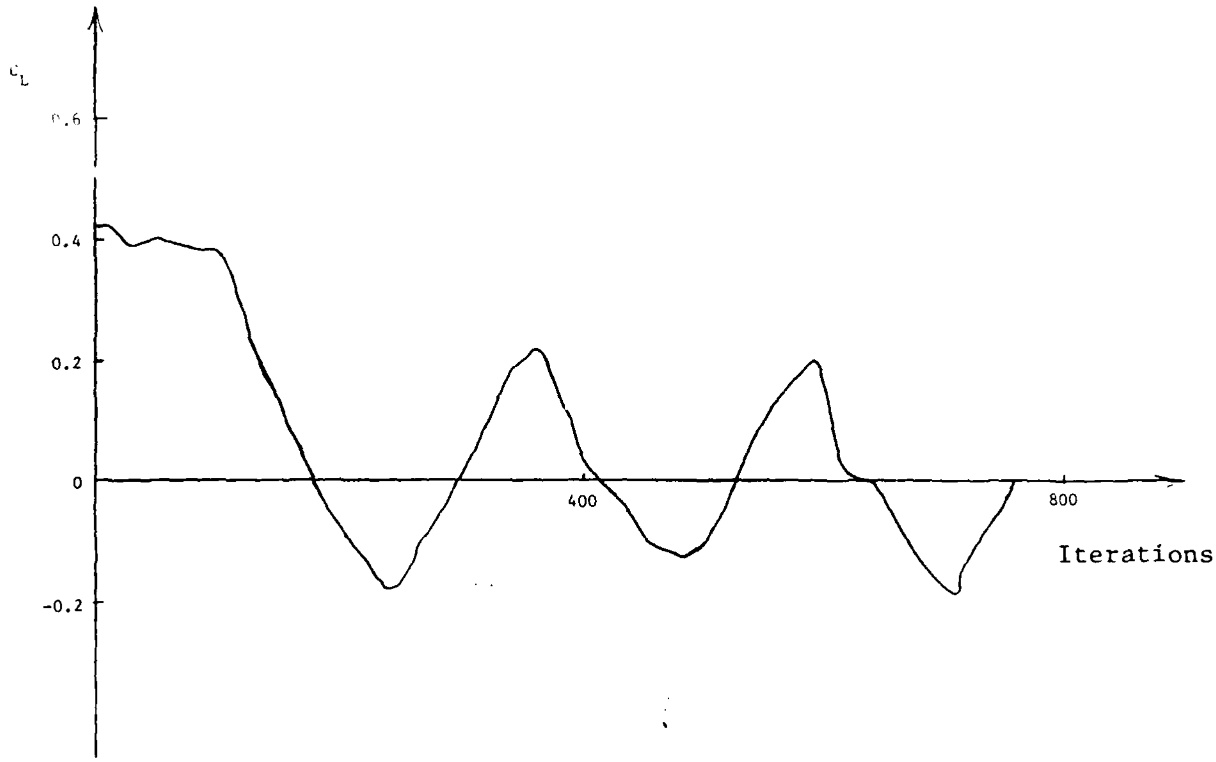

Figure 11.- Trace of C_L vs. time for a 14% biconvex airfoil, $M_\infty = 0.81$, $\alpha = 0.0$ deg; TSD equation with viscous wedge.